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Stability Analysis of Prey-Predator Population Model with Harvesting on The Predator Population

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Abstract

In this paper we present a deterministic and continuous model for one prey–one predator population model based on Lotka-Volterra model. The predator population is subjected to both constant effort and constant quota of harvesting. We study analytically the sufficient conditions of harvesting to ensure the stability of the equilibrium point. The method used to analyze the stability of the equilibrium point is linearization and Hurwitz stability test. The results show that the equilibrium point which occurs in positive quadrant is stable although the predator population is subjected to harvesting. This means that the prey and predator populations can live in coexistence although the predator is harvested provided the level of harvesting is controlled. Some examples are given to illustrate the behavior of the trajectories.

Kata Kunci: Prey-predator, Stability, Harvesting, Constant quota, Effort.

1. Introduction

The prey-predator model based on Lotka-Volterra model is one of the most popular equations in mathematical ecology. The results of some authors on Lotka-Volterra model can be found in (Luckinbill, 1973). Their conclusions indicate that the prey and predator population can coexist by reducing the frequency of contact between them. In Danca *et al.* (1997), they have analyzed a prey-predator model using analytical and numerical methods. They found that the system can exhibit a rich behavior and determined the domain of the value of the parameters for which the system has stationary states or chaotic behavior.

In Brauer & Soudack (1979a, 1979b, 1981), they have considered some general prey-predator models which include harvesting problems at a constant rate. They analyzed the global behavior of the prey-predator model and classified the possibilities and also determined the domain of attraction of the trajectory. They also found that under certain conditions the model with harvesting is stable. When the prey-predator model enjoys an asymptotically stable condition and the stability is weak then an asymptotically limit cycle will probably exist (Jeffries, 1974).

In Kar & Chaudhuri (2004), they have studied the prey-predator model based on Lotka-Volterra model with harvesting. They discussed about the possibility of existence of bionomic equilibrium and optimal harvesting. The stability of effect constant quota and effort constant has been studied by Holmberg (1995) and he showed that constant catch quota can lead to both oscillations and chaos and an increased risk for over exploitation.

A prey-predator model with Holling type using harvesting effort s as control has been presented by Srinivasu *et al*. (2001). He showed that with harvesting is possible to break the cyclic behavior of the system and introduce globally stable limit cycle in the system.

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In this paper we present a deterministic and continuous prey-predator population model based on Lotka-Volterra model. The predator population is harvested with constant effort and constant quota of harvesting. The stability of the equilibrium point and the effect of harvesting are investigated. Some examples are given to show the behavior of the trajectories around the equilibrium points. The Maple software is used to plot the trajectories and direction field of the model.

2. Prey-Predator Model without Harvesting

We consider a prey-predator model based on Lotka-Volterra model with one-prey and one-predator populations. The models for the rate of change of prey population (x) and predator population (y) with respect to time t are as follows

$$\begin{aligned} \dot{x} &= ax\left(1-\frac{x}{K}\right)-\alpha xy \\ \dot{y} &= -cy+\beta xy. \end{aligned} \tag{1}$$

The model includes parameter K, the carrying capacity, of the prey population in the absence of the predator. Parameter a is the intrinsic growth rate of prey, c is the mortality rate of the predator without prey, α measures the rate of consumption of prey by the predator, and β measures the conversion of prey consumed into the predator reproduction rate. We assume all parameters are real positive numbers.

For simplification model (1) is written on the form

$$\begin{aligned} \dot{x} &= x(a-bx-\alpha y) \\ \dot{y} &= y(-c+\beta x). \end{aligned} \tag{2}$$

The possible positive equilibrium point of this model is $E^* = (x^*, y^*) = \left(\frac{c}{\beta}, \frac{a\beta - bc}{\alpha\beta}\right)$ when $a\beta - bc > 0$. Jacobian matrix of (2) takes the form

$$J = \begin{pmatrix} a-2bx-\alpha y & -\alpha x \\ \beta y & -c+\beta x \end{pmatrix}, \tag{3}$$

and at E^*, we have

$$J = \begin{pmatrix} -\frac{bc}{\beta} & -\frac{\alpha c}{\beta} \\ \frac{a\beta - bc}{\alpha} & 0 \end{pmatrix}.$$

The characteristic function of Jacobian matrix J at this point is

$$f(r)=\begin{vmatrix} r+\dfrac{bc}{\beta} & \dfrac{\alpha c}{\beta} \\ \dfrac{-(a\beta-bc)}{\alpha} & r \end{vmatrix}=r^2+\frac{bc}{\beta}r+\frac{c}{\beta}(a\beta-bc).$$

The eigenvalues are $r_{1,2}=\dfrac{-P\pm\sqrt{P^2-4Q}}{2}$, where $P=\dfrac{bc}{\beta}$ and $Q=\dfrac{c}{\beta}(a\beta-bc)$. Since P and Q are both positive numbers, then both eigenvalues have negative real parts. Then the equilibrium point E^* is locally asymptotically stable. However, since $a\beta-bc>0$ the equilibrium point E^* of the model (2) is also globally asymptotically stable as stated in the theorem by Chao & Yuei (2002).

3. Prey-Predator Model with Constant Effort of Harvesting

The model (2) is improved by considering harvesting with a constant effort on the predator population. The harvesting function is proportional the population size of predator. The model becomes

$$\begin{aligned}\dot{x}&=x(a-bx-\alpha y)\\ \dot{y}&=y(-c+\beta x)-Ey\ ,\end{aligned}\tag{4}$$

where E is a positive constant effort.

The model (4) is mathematically similar to the model (2). The model (4) possibly has a positive equilibrium point

$$E_H^*=(x^*,y^*)=\left(\frac{c+E}{\beta},\frac{a\beta-b(c+E)}{\alpha\beta}\right),$$

when $0<E<\dfrac{a\beta-bc}{b}$. By following the similar procedure as before, we conclude that the equilibrium point E_H^* of model (4) is also globally asymptotically stable, whenever the effort of harvesting satisfies the condition $0<E<\dfrac{a\beta-bc}{b}$.

4. Prey-Predator Model with Constant Quota of Harvesting

We consider the model (2) where the predator population is now subjected to harvesting at constant quota. The model becomes

$$\begin{aligned}\dot{x}&=x(a-bx-\alpha y)\\ \dot{y}&=y(-c+\beta x)-H\ ,\end{aligned}\tag{5}$$

where H is a positive constant quota of harvesting.

The possible positive equilibrium point of model (5) is (x^*, y^*), where $x^* = \dfrac{a - \alpha k}{b}$, $y^* = k$, and k is the root of quadratic equation $\alpha\beta Z^2 - (a\beta - bc)Z + Hb = 0$. The Jacobian matrix for this model still refers to (3).

Let $A = a - 2bx - \alpha y$, $B = \alpha x$, $C = \beta y$, and $D = -c + \beta x$, then the characteristic function from the Jacobian matrix (3) can be written in the form

$$r^2 - (A + D)r + (AD + BC) = 0. \qquad (6)$$

From equation (6) we write $p_0 = AD + BC$ and $p_1 = -(A + D)$.

Let $P = \alpha\beta$, $Q = a\beta - bc$, and $R = Hb$. Assume that $a\beta - bc > 0$ and $Q^2 - 4PR \geq 0$.

<u>Case 1</u>. If $Q^2 = 4PR$; that is $H = \dfrac{(a\beta - bc)^2}{4\alpha\beta b}$.

The equilibrium point for model (2) is $\left(\dfrac{a\beta + bc}{2b\beta}, \dfrac{a\beta - bc}{2\alpha\beta}\right)$.

It follows that the determinant of the Jacobian matrix at this point is zero so that linearization method cannot be used to analyze the stability of the equilibrium point. By using phase plane analysis, we found that the equilibrium point is not stable.

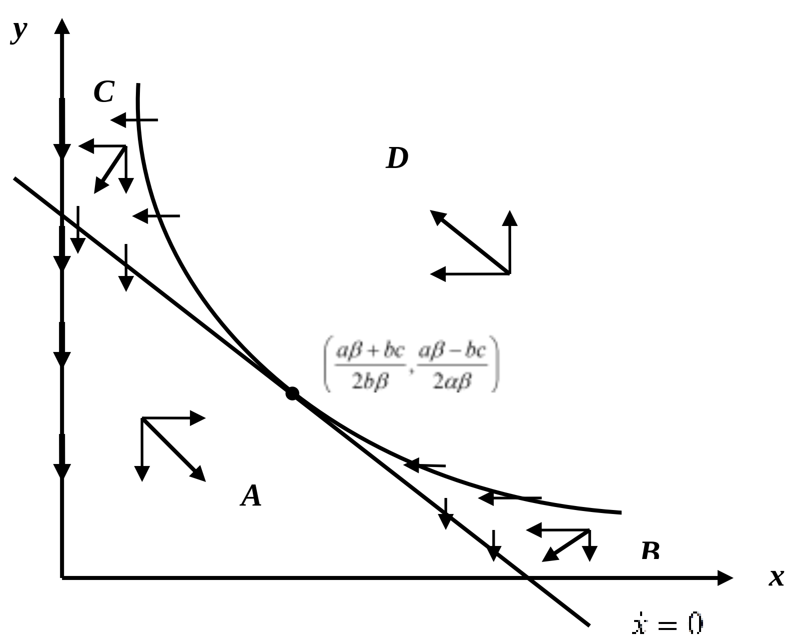

Figure 1. Phase plane for case1.

When $(x(0), y(0))$ is in the region ***D***, as time passes the trajectory of $(x(t), y(t))$ will enter the region ***C*** and then goes to the region ***A***. Finally the predator population $y(t)$ becomes extinct. While when the initial population $(x(0),\ y(0))$ is in the region ***B***, then the trajectory of $(x(t),\ y(t))$ will enter the region ***A*** and finally the predator population $y(t)$ also becomes extinct.

<u>Case2</u>. If $Q^2 - 4PR > 0$; that is, $H < \dfrac{(a\beta - bc)^2}{4\alpha\beta b}$.

In this case there are two equilibrium points namely $E_1^* = (x_1^*,\ y_1^*)$ and $E_2^* = (x_2^*,\ y_2^*)$, where

$$x_1^* = \frac{a - \alpha y_1^*}{b},\ x_2^* = \frac{a - \alpha y_2^*}{b},\ y_1^* = \frac{Q + \sqrt{Q^2 - 4PR}}{2P},\ \text{and}\ y_2^* = \frac{Q - \sqrt{Q^2 - 4PR}}{2P}.$$

Lemma 1.

Let $H < \dfrac{(a\beta - bc)^2}{4\alpha\beta b}$ *and* $E_1^* = (x_1^*,\ y_1^*)$ *be the equilibrium point for model* (5). *The determinant of the Jacobian matrix J at* E_1^* *is positive.*

Proof: The determinant of the Jacobian matrix *J* at E_1^* is

$$\det(J) = AD + BC = -ac + a\beta x_1^* + 2bcx_1^* - 2b\beta(x_1^*)^2 + \alpha c y_1^*.$$

Substitute $y_1^* = \dfrac{a - bx_1^*}{\alpha}$ into the above equation we obtain $\det(J) = x_1^*(a\beta + bc - 2b\beta x_1^*)$.

From the term $Q^2 - 4PR > 0$ then the following inequality is valid

$$Q < Q + \sqrt{Q^2 - 4PR} \qquad \text{or} \qquad a\beta - bc < 2\alpha\beta\left(\frac{Q + \sqrt{Q^2 - 4PR}}{2P}\right).$$

Since $y_1^* = \dfrac{Q + \sqrt{Q^2 - 4PR}}{2P}$, the inequality can be written as $a\beta - bc < 2\alpha\beta y_1^*$. Hence $2a\beta - 2\alpha\beta y_1^* < a\beta + bc$, or $\dfrac{a - \alpha y_1^*}{b} < \dfrac{a\beta + bc}{2b\beta}$. Since $x_1^* = \dfrac{a - \alpha y_1^*}{b}$, we have $(a\beta + bc - 2b\beta x_1^*) > 0$, or equivalently

$$x_1^*(a\beta + bc - 2b\beta x_1^*) > 0. \tag{7}$$

Therefore, from (7) we conclude that $\det(J) > 0$ or $p_0 > 0$.□

Theorem 1.

Let $H < \dfrac{(a\beta - bc)^2}{4\alpha\beta b}$. *Then the equilibrium point* $E_1^* = (x_1^*,\ y_1^*)$ *is an attractor trajectory, if*

i) $b = \beta$, *or*

ii) $b < \beta$ *and* $a(\beta - b) - bc \le 0$ *, or*
iii) $b < \beta$ *and* $a\beta(\beta - b) - bc\beta - b^2 c \le 0$ *, or*
iv) $b < \beta$ *and* $H < \dfrac{(a\beta - ab - bc)bc}{\alpha(b-\beta)^2}$.

Note that $tr(J) = A + D$ and $p_1 = -(A + D)$. We have $A + D = a - 2bx_1^* - \alpha y_1^* - c + \beta x_1^*$. Substitute $x_1^* = \dfrac{a - \alpha y_1^*}{b}$ to get $A + D = \dfrac{a\beta - ab - bc + (b - \beta)\alpha y_1^*}{b}$ and also substitute $y_1^* = \dfrac{Q + \sqrt{Q^2 - 4PR}}{2P}$ into the numerator so that the numerator is on the form

$$a\beta - ab - bc + (b - \beta)\left(\frac{Q + \sqrt{Q^2 - 4PR}}{2\beta}\right). \qquad (8)$$

In order to determine the sign of $tr(J) = A + D$, we will consider the sign of (8).

Proof of Theorem 1: Since $b = \beta$, then (8) is negative. It follows that $tr(J) = A + D$ is negative or $p_1 > 0$.

i) Since $b < \beta$ and $a(\beta - b) - bc \le 0$, it follows (8) is negative and $p_1 > 0$. We consider $a\beta(\beta - b) - bc\beta - b^2 c$. This form can be written as

$$2\beta[a(\beta - b) - bc] - (a\beta - bc)(\beta - b).$$

Since $b < \beta$ and $a\beta(\beta - b) - bc\beta - b^2 c \le 0$, then we have the inequality $\dfrac{2\beta[a(\beta - b) - bc]}{(\beta - b)} - (a\beta - bc) \le 0$. After some manipulations we get $a\beta - ab - bc + (b - \beta)\left(\dfrac{Q + \sqrt{Q^2 - 4PR}}{2\beta}\right) < 0$. This follows that $p_1 > 0$.

ii) The condition $H < \dfrac{(a\beta - ab - bc)bc}{\alpha(b-\beta)^2}$ can be written in the form $H < \dfrac{-4(ab - a\beta + bc)b^2\beta c}{4\alpha\beta b(b-\beta)^2}$, i.e. $H < \dfrac{2QG - G^2}{4\alpha b\beta}$, where $G = \dfrac{2\beta[a(\beta - b) - bc]}{(\beta - b)}$.
After some algebraic manipulations, the inequality becomes $\dfrac{2\beta[a(\beta - b) - bc]}{(\beta - b)} - \left(Q + \sqrt{Q^2 - 4PR}\right) < 0$.

Since $b < \beta$, we have $a\beta - ab - bc + (b - \beta)\left(\dfrac{Q + \sqrt{Q^2 - 4PR}}{2\beta}\right) < 0$.

This follows that $p_1 > 0$.

Therefore, since $p_1 > 0$ and $p_0 > 0$ (Lemma 1), we conclude that if the condition i) or ii) or iii) or iv) is satisfied and following the Hurwitz stability test (Willems, 1970; Jeffries, 1989) the equilibrium point E_1^* is an attractor trajectory. □

Theorem 2.
The equilibrium point $E_1^* = (x_1^*, y_1^*)$ *is an attractor trajectory if the conditions* $H < \dfrac{(a\beta - bc)^2}{4\alpha\beta b}$ *and* $b > \beta$ *are satisfied.*

Proof: Since $b > \beta$, the form $\dfrac{-4(ab - a\beta + bc)b^2\beta c}{4\alpha\beta b(b-\beta)^2} < 0$. Because $H > 0$ then we have an inequality $H > \dfrac{-4(ab - a\beta + bc)b^2\beta c}{4\alpha\beta b(b-\beta)^2}$. The inequality can be expressed as $H > \dfrac{2QG - G^2}{4\alpha b\beta}$, where $G = \dfrac{2\beta[a(\beta - b) - bc]}{(\beta - b)}$. Then we get $4\alpha\beta Hb > 2QG - G^2$. Furthermore $Q^2 - 4PR < (G-Q)^2$. It is easy to show that $G - Q > 0$, then the inequality can be written in the form $\sqrt{Q^2 - 4PR} < G - Q$, or $\dfrac{2\beta[a(\beta - b) - bc]}{(b-\beta)} + \left(Q + \sqrt{Q^2 - 4PR}\right) < 0$. Since $b > \beta$, then we have $a\beta - ab - bc + (b-\beta)\left(\dfrac{Q + \sqrt{Q^2 - 4PR}}{2\beta}\right) < 0$. It follows that $p_1 > 0$. Therefore, since $p_1 > 0$ and $p_0 > 0$ (Lemma 1), we conclude that the equilibrium point E_1^* is an attractor trajectory. □

From the two cases we know that the equilibrium E_1^* may be a stable or an unstable equilibrium point. It depends on the values of the parameters and the harvesting function.

Apparently, the equilibrium point E_1^* tends to the equilibrium point $\left(\dfrac{c}{\beta}, \dfrac{a\beta - bc}{\alpha\beta}\right)$ when the harvesting function *H* approaches zero. If the equilibrium point $\left(\dfrac{c}{\beta}, \dfrac{a\beta - bc}{\alpha\beta}\right)$ for the non-harvesting model, model (2), is asymptotically stable, the eigenvalues of the Jacobian matrix of the linearized system have negative real part. Since the eigenvalues are continuous in *H*, the equilibrium point E_1^* is asymptotically stable for sufficiently small $H > 0$. On the other hand, if the equilibrium point E_1^* is unstable, so that there is an asymptotically stable limit cycle, then the theory of perturbation of periodic solutions (Coddington & Levinson, 1955) shows that there is an asymptotically stable limit cycle for small $H > 0$. Thus, the qualitative behavior of the system for $H = 0$ carries over to small $H > 0$ (Brauer & Soudack, 1979b).

Lemma 4.

Let $H < \frac{(a\beta - bc)^2}{4\alpha\beta b}$, *and* $E_2^* = (x_2^*, y_2^*)$ *be an equilibrium point for model* (5). *Then the determinant of the Jacobian matrix J at* E_2^* *is negative.*

Proof: The determinant of the Jacobian matrix J associated with the equilibrium point E_2^* is

$$\det(J) = AD + BC = -ac + a\beta x_2^* + 2bcx_2^* - 2b\beta (x_2^*)^2 + \alpha c y_2^* .$$

Substitute $y_2^* = \frac{a - bx_2^*}{\alpha}$ into the above equation to get $\det(J) = x_2^*(a\beta + bc - 2b\beta x_2^*)$. We show that $\det(J) < 0$ by contradiction. Let $\det(J) \geq 0$, that is $x_2^*(a\beta + bc - 2b\beta x_2^*) \geq 0$. Since $H < \frac{(a\beta - bc)^2}{4\alpha\beta b}$, the two equilibrium points are positive and different, thus $x_2^* > 0$. So we have $(a\beta + bc - 2b\beta x_2^*) \geq 0$, or $x_2^* \leq \frac{a\beta + bc}{2b\beta}$. Since $x_2^* = \frac{a - \alpha y_2^*}{b}$, the inequality becomes $\frac{a - \alpha y_2^*}{b} \leq \frac{a\beta + bc}{2b\beta}$, or $a\beta - bc \leq 2\alpha\beta y_2^*$. Substitute $y_2^* = \frac{Q - \sqrt{Q^2 - 4PR}}{2P}$ into the above inequality to obtain $a\beta - bc \leq 2\alpha\beta\left(\frac{Q - \sqrt{Q^2 - 4PR}}{2\alpha\beta}\right)$, or $\sqrt{Q^2 - 4PR} \leq 0$. This contradicts the fact that $\sqrt{Q^2 - 4PR} > 0$. Therefore we conclude that $\det(J) < 0$.□

Theorem 3.

Let $H < \frac{(a\beta - bc)^2}{4\alpha\beta b}$. *Then the equilibrium point* $E_2^* = (x_2^*, y_2^*)$ *is a saddle point.*

Proof: Since the determinant of the Jacobian matrix associated with this equilibrium point is negative (Lemma 4), it follows that the eigenvalues of the Jacobian matrix are real with different signs. This means that the equilibrium point E_2^* is a saddle point. □

5. Some Examples

In this section, we illustrate our result by some examples. Graph of the trajectories for non linear model are plotted.

Example 1. Consider model (5) with parameters $a = 1$, $b = 0.04$, $\alpha = 1$, $c = 0.1$ $\beta = 0.05$, $H = 0.27$**. There is no equilibrium point since** $H > \dfrac{(a\beta - bc)^2}{4\alpha\beta b}$**. Some trajectories of** $(x(t),\ y(t))$ **are given in Figure 2. The predator population** $y(t)$ **will become extinct.**

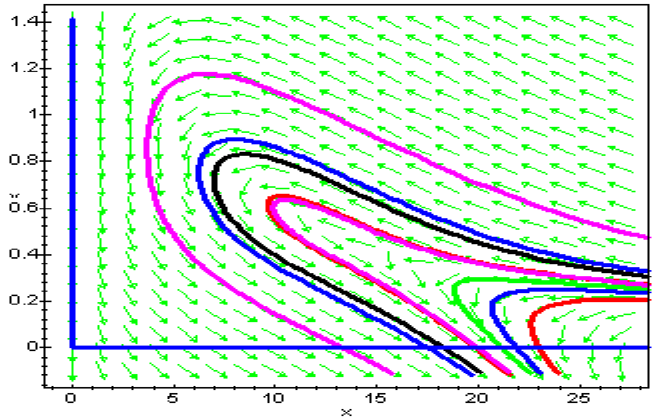

Figure 2. Some trajectories of $(x(t),\ y(t))$**, where the predator population will be extinct.**

Example 2. Consider model (5) with parameters $a = 1$, $b = 0.04$, $\alpha = 1$, $c = 0.3$ $\beta = 0.5$, **and** $H = 1.728$**. There is only one equilibrium point, i.e.** $(13.00,\ 0.48)$ **since** $H = \dfrac{(a\beta - bc)^2}{4\alpha\beta b}$**. Some trajectories of** $(x(t),\ y(t))$ **are given in figure 3. The trajectories finally tend away from the equilibrium point. The predator population** $y(t)$ **will become extinct.**

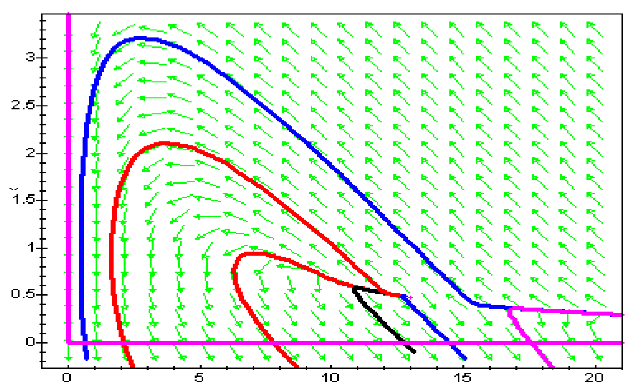

Figure 3. The trajectories of $(x(t),\ y(t))$ **keep away from the unstable equilibrium point.**

Example 3. Consider model (5) with parameters $a=1,\ b=0.04,\ \alpha=1,\ c=0.3\ \beta=0.05,$ **and** $H=0.0805.$ **There are two equilibrium points i.e.** $E_1^*=(8.4289\,,\,0.6628\,)$ **and** $E_2^*=(22.5711\,,\,0.9716\,)$ **since** $H<\dfrac{(a\beta-bc)^2}{4\alpha\beta b}$**. The eigenvalues associated with the equilibrium point** E_1^* **for linear model are** $-0.1079\pm0.4762i$**. Some trajectories of** $(x(t),\ y(t))$ **are given in figure 4. Some of trajectories tend spirally to the locally stable** E_1^***. If** $(x(0),\ y(0))$ **is close enough to** E_1^***, the predator and prey populations will continue to exist although the predator population is harvested.**

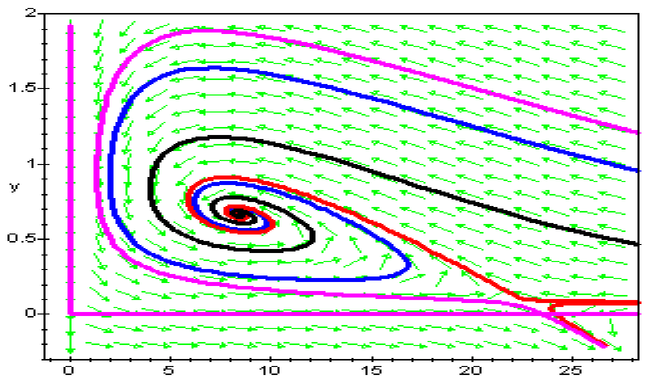

Figure 4. Some trajectories of $(x(t),\ y(t))$ **tend to the local stable equilibrium point.**

Example 4. Consider model (5) with parameters $a=1,\ b=0.04,\ \alpha=1,\ c=0.1\ \beta=0.05,$ **and** $H=0.2390$ **There are two equilibrium points i.e.** $E_1^*=(9.9293\,,\,0.6028\,)$ **and** $E_2^*=(17.0707\,,\,0.3172\,)$ **since** $H<\dfrac{(a\beta-bc)^2}{4\alpha\beta b}$**. The eigenvalues associated with the equilibrium point** E_1^* **for linear model are** $-0.0003\pm0.3766i$ **.Some trajectories of** $(x(t),\ y(t))$ **are given in figure 5. Some of trajectories tend spirally to the locally stable** E_1^***. There are trajectories move spirally around the** E_1^* **but finally tend away from it.**

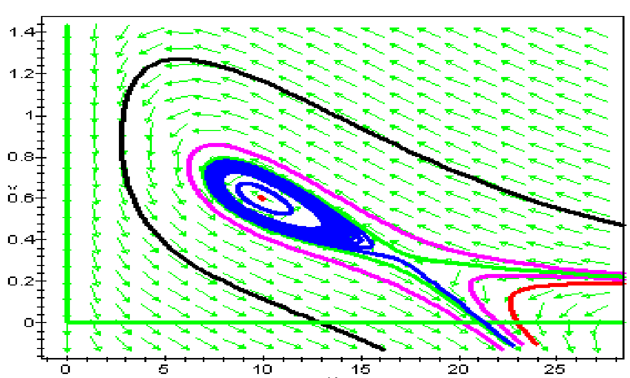

Figure 5. Trajectories of $(x(t),\ y(t))$ around the local stable equilibrium point.

Example 5. Consider model (5) with parameters $a=1,\ b=0.04,\ \alpha=1,\ c=0.1\ \beta=0.05,$ and $H=0.2600$ There are two equilibrium points i.e. $E_1^*=(12.0000\,,\,0.5200)$ and $E_2^*=(15.0000\,,\,0.4000)$ since $H<\dfrac{(a\beta-bc)^2}{4\alpha\beta b}$. Although the two equilibrium points exist, the conditions do not satisfy theorem 2.iv, i.e. $H>\dfrac{(a\beta-ab-bc)bc}{\alpha(b-\beta)^2}$, the two equilibrium points are not stable. The eigenvalues associated with the equilibrium point E_1^* for linear model are $0.0100\pm0.2681i$. Some trajectories of $(x(t),\ y(t))$ are given in figure 6. The trajectories move around the two equilibrium points and then tend away from the equilibrium points. The predator population $y(t)$ will become extinct.

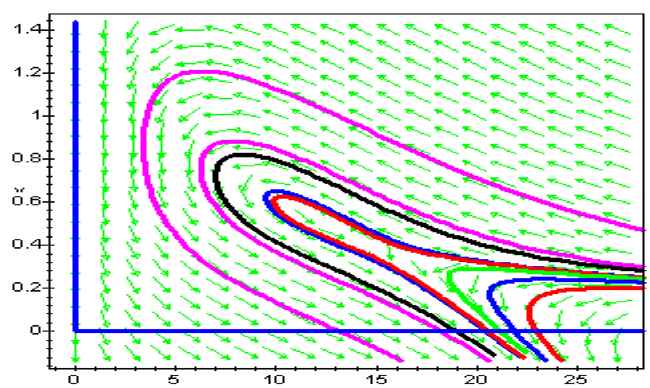

Figure 6. The trajectories of $(x(t),\ y(t))$ move and then tend away from two unstable equilibrium points.

9. Conclusions

From the analysis of the prey-predator model, we found that model without harvesting has a globally stable equilibrium point, when $a\beta-bc>0$. It means that the two populations may live in coexistence. When the predator population is harvested with constant effort of harvesting satisfying $0<E<\dfrac{a\beta-bc}{b}$, the two populations remain stable.

If the predator population is harvested with constant quota of harvesting, the model also possibly has a locally stable equilibrium point. This situation occurs when the parameters of the model and harvesting level are strictly controlled, Theorem 1 and 2.

The coexistence of both populations depends on the considered values of the parameters, the level of harvesting, and the initial size of the populations. When the initial size of the populations is too far from the stable equilibrium point, the trajectories may not tend to the stable equilibrium point as time passes since we just consider local stability of the equilibrium point.

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